Cube Root Extraction by Long Division: A Unified Formula for All Real Numbers

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Abstract

This paper presents a systematic long division method for computing cube roots of any real number. The method applies uniformly to whole numbers, decimals, perfect cubes, and non-perfect cubes, achieving arbitrary precision through a single generalized formula. Unlike traditional approaches, this technique supports flexible digit grouping (similar to standard long division) while maintaining consistency across all cases. Examples include $\sqrt[3]{15069223} = 247$ for perfect cubes and $\sqrt[3]{1218.2161} \approx 10.680087$ for decimal cases.

Keywords: Cube root calculation, long division method, digit grouping algorithm, root extraction, manual computation, numerical methods, arbitrary precision arithmetic

1 Introduction

Computing cube roots manually has largely been abandoned in favor of calculator algorithms and iterative approximations like Newton's method [1]. Although these approaches excel computationally, they offer little intuition for hand-calculated situations or educational contexts. Classical digit-by-digit extraction methods exist, but remain obscure and limited to specific number formats.

This paper presents a systematic long division algorithm for cube roots that works uniformly across all real numbers - whole, decimal, perfect, and non-perfect cubes. The key innovation lies in its flexible digit grouping mechanism, analogous to standard division, which maintains algorithmic consistency regardless of input format. The method achieves arbitrary precision through the iterative application of a single generalized formula, which makes it both theoretically complete and practically accessible for manual computation.

2 Methodology

This section presents a unified long division algorithm for computing cube roots of any real number. The method extends the classical extraction of digits by digits through a generalized remainder formula that handles variable-length digit grouping.

2.1 Core Formula

The algorithm operates on a pattern function that computes the contribution of each candidate to the working remainder.

$$f(c, q, q') = \left(10^k \cdot 3 \cdot c \cdot q \cdot q'\right) + c^3$$

where:

- c is the candidate (integer or decimal),
- q is the current quotient,
- k is the number of digits in the integer part of the candidate c. Note: The leading zeros of the candidate are counted.
- q' is the updated quotient after incorporating the candidate c :

$$q' = (q \cdot 10^k) + c$$

2.2 Digit Partitioning

The input number is partitioned into chunks of three digits, working outward from the decimal point:

- **Integer part**: Partition from right to left (toward the most significant digit). The leftmost partition may contain fewer than three digits (Can be padded with leading zeros).
- **Decimal part**: Partition from left to right (toward lower precision). The rightmost partition is padded with trailing zeros if incomplete.

Example: For 1218216.1247:

$$\underbrace{\underbrace{1}_{\text{Partition 1}}}_{\text{Group 1}} \;\Big|\; \underbrace{\underbrace{218}_{\text{Partition 2}}}_{\text{Group 2}} \;\Big|\; \underbrace{\underbrace{216}_{\text{Partition 3}} \;.\; \underbrace{124}_{\text{Partition 4}}}_{\text{Group 3}} \;\Big|\; \underbrace{\underbrace{700}_{\text{Partition 5}}}_{\text{Group 4}}$$

The number is first partitioned as:

- **Partition 1**: 1
- **Partition 2**: 218
- **Partition 3**: 216
- **Partition 4**: 124
- **Partition 5**: 700

These partitions may be flexibly grouped into computational units as needed:

- **Group 1**: Partition 1 (1)
- **Group 2**: Partition 2 (218)
- **Group 3**: Partitions 3 and 4 combined (216.124)
- **Group 4**: Partition 5 (700)

A **group** is defined as one or more consecutive partitions brought down simultaneously. For instance, partitions 2 and 3 can be combined as a group “218216”, or partition 3 and 4 can be combined as a group “216.124“ or partition 4 can be processed alone as a group “124”. The minimum unit is one complete partition - partial digits cannot be bring down independently.

2.3 Candidate Classification and Formula Selection

Table 1 demonstrates how to determine the appropriate value of k based on candidate structure.

$\sqrt[3]{225866.529} = 60.9$, we can consider different case of grouping as below:

- **Case 1:** 225|866.529

$$\underbrace{\underbrace{225}_{\text{Partition 1}}}_{\text{Group 1}} \;\Big|\; \underbrace{\underbrace{866}_{\text{Partition 2}} \;.\; \underbrace{529}_{\text{Partition 3}}}_{\text{Group 2}}$$

- **Case 2:** 225866|.529

$$\underbrace{\underbrace{225}_{\text{Partition 1}} \; \underbrace{886}_{\text{Partition 2}}}_{\text{Group 1}} \;\Big|.\; \underbrace{\underbrace{529}_{\text{Partition 3}}}_{\text{Group 2}}$$

As part of the algorithm,

- **Case 1:** The candidate C_1 will be 6 and C_2 will be 0.9
- **Case 1:** The candidate C_1 will be 60 and C_2 will be .9

For C_2 the first case contains the integer part of 0 in the candidate, Therefore:

- In Case 1: For C_2 we use $k = 1$ and $q' = (q \cdot 10^k) + c = 10q + c$

- In Case 2: For C_2 we use $k = 0$ and $q' = (q \cdot 10^k) + c = q + c$

In the second case the candidate is $c = .6$, where it doesn't have an integer part, Hence $k = 0$.

Selection Rule: k = Number of digits in the integer part of the candidate. Example: $c = 12.3$, This candidate have integer part 12, So $k = 2$ in this case.

Table 1: k value , based on different cases of candidate

Candidate c (when)	**k value**	**Case Type**
0	1	Zero (single digit)
1	1	Single integer digit
12	2	Two integer digits
121	3	Three integer digits
00	2	Zero with leading zero preserved
00122	5	Zero with leading zero preserved
.1	0	One decimal place
.12	0	Two decimal places
.132	0	Three decimal places
.10	0	Decimal with trailing zero preserved
1.2	1	Mixed (Single integer digit & Single decimal digit)
1.23	1	Mixed (Single integer digit & Multiple decimal digits)
12.3	2	Mixed (Multiple integer digits & 1 decimal digit)
0.0	1	Only Zeros
0.00	1	Only Zeros
.00	0	Only Zeros - Pure decimal candidate
00.012	2	In this case we are bringing 5 partitions as part of a single group, and since integer part of candidate takes priority, Hence value of k determine accordingly
0.6	1	In this case as part of a single group we have two partitions, and in 0.6, the 0 belongs to the integer part of candidate. Refer above.
.6	0	In this case the as part of a single group we have one partition, Hence value of q' & k determine accordingly. Refer above.

2.4 Algorithmic Procedure

1. **Partition of digits**: Divide the input number into partitions of three digits, working outward from the decimal point. For the integer part, partition from right to left; the leftmost partition may contain fewer than three digits. For the decimal part, partition from left to right, padding the rightmost partition with trailing zeros if incomplete. For example, 1218216.1247 partitions as $1\,|\,218\,|\,216\,|\,.124\,|\,700$. A **group** consists of one or more consecutive partitions brought down together (e.g., "218" alone, or "218216" combined).

2. **Initialize with the leftmost partition**: Let n_1 be the leftmost partition. Find the largest value c_1 such that $c_1^3 \leq n_1$. Since $q_0 = 0$ initially, the formula simplifies to:

$$f(c_1, 0, q_1) = (10^k \cdot 3 \cdot c_1 \cdot 0 \cdot q_1) + c_1^3 = c_1^3$$

where k equals number of digits in the integer part of c_1 with leading zeros preserved and $q_1 = c_1$ (since $q_0 = 0$). Set quotient $q = c_1$ and remainder $r = n_1 - c_1^3$.

3. **Bring down the next group**: Select one or more partitions to form the next group (minimum one partition). Append this group to remainder r to create working number n'. For instance, if $r = 1$ and group "218" is brought down, then $n' = 1218$. k will be determine by based on the value of candidate.

4. **Determine the candidate**: Find the largest value c satisfying:

$$f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq n'$$

where q' is :

$$q' = (q \cdot 10^k) + c$$

where k is the number of digits in the integer part of the candidate c with leading zeros preserved.

If no valid $c \geq 1$ exists (i.e., $n' < f(1, q, q')$), use $c = 0$, which yields $f(0, q, q) = 0$.

Update remainder $r = n' - f(c, q, q')$ and quotient $q = q'$.

Note: The number of digits in the candidate must be equal to the size of the group, and it should correspond with the decimal placements, where leading and trailing zeros can be padded

5. **Iterate or terminate**: Repeat steps 3 and 4 for each subsequent group. Adjust group size (number of partitions) as desired - larger groups accelerate computation but require broader candidate searches. If all partitions are exhausted and $r = 0$, the quotient q is exact.For non-perfect cubes or extended precision, append additional partitions of zeros (3 zeros per partition) and continue iteration.

6. **Output**: The final quotient q represents the cube root of the computed precision, with integer and decimal components determined by partition placement.

This algorithm applies formula $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3$ uniformly across all iterations. The parameter k adapts to the number of digits in the integer part of the candidate with the leading zeros preserved, while $q' = (q \cdot 10^k) + c$. This unified approach handles for all real numbers without case-specific modifications, with grouping flexibility providing computational efficiency control.

3 Examples

We demonstrate the cube root long division algorithm through few cases that illustrate how the method handles different input formats. To emphasize the consistency of the algorithm, we use the same underlying value presented in different ways.

The examples are:

1. **Perfect Cube Case**: Computing the exact cube root of an integer that is a perfect cube
2. **Non-perfect Cube Case**: Computing the cube root of an integer to arbitrary decimal precision
3. **Decimal Radicand Case**: Computing the cube root of a decimal number

Each example follows the same procedure, showing how the digit grouping adapts to different formats while the computational steps remain unchanged.

3.1 Perfect Cube Case

This subsection demonstrates the algorithm on a perfect cube where the result is an exact integer. We use $\sqrt[3]{58495135018 32} = 18018$ as our example. The same number is computed multiple times with different digit groupings to show that the algorithm produces identical results regardless of the grouping choice.

3.1.1 Standard Grouping: $5\,|\,849\,|\,513\,|\,501\,|\,832$

In this example,We partition the digits as follows, and group the partitions as below:

$$\underbrace{\underbrace{5}_{\text{Partition 1}}}_{\text{Group 1}} \;|\; \underbrace{\underbrace{849}_{\text{Partition 2}}}_{\text{Group 2}} \;|\; \underbrace{\underbrace{513}_{\text{Partition 3}}}_{\text{Group 3}} \;|\; \underbrace{\underbrace{501}_{\text{Partition 4}}}_{\text{Group 4}} \;|\; \underbrace{\underbrace{832}_{\text{Partition 5}}}_{\text{Group 5}}$$

1. **Group the digits**: Split 5,849,513,501,832 into partition by digit partition rule and group into computational units: $5\,|\,849\,|\,513\,|\,501\,|\,832$ (or $005\,|\,849\,|\,513\,|\,501\,|\,832$, where the leftmost & rightmost group is padded as needed).

2. **Initialize with the leftmost group**: With $n' = 5$ and $q = 0$, find the largest c_1 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 5$

- If $c_1 = 0$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 0 = 0$,

$$f(0,0,0) = (10^1 \cdot 3 \cdot 0 \cdot 0 \cdot 0) + 0^3 = 0 \leq 5$$

- If $c_1 = 1$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 1 = 1$,

$$f(1,0,1) = (10^1 \cdot 3 \cdot 1 \cdot 0 \cdot 1) + 1^3 = 1 \leq 5$$

- If $c_1 = 2$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 2 = 2$,

$$f(2,0,2) = (10^1 \cdot 3 \cdot 2 \cdot 0 \cdot 2) + 2^3 = 8 > 5$$

$$\begin{array}{rccccccccccccc}
 & 1 & & & & & & & & & & & & \\ \cline{2-14}
1) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
 & 1 & & & & & & & & & & & & \\ \cline{2-2}
 & 4 & & & & & & & & & & & &
\end{array}$$

Choose $c_1 = 1$. Set $q = 1$, remainder $r = 5 - 1 = 4$.

3. **Bring down the next group**: Bring the next group 849 and append to $r = 4$, forming $n' = 4849$.

$$\begin{array}{rccccccccccccc}
 & 1 & & & & & & & & & & & & \\ \cline{2-14}
1) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
 & 1 & & & & & & & & & & & & \\ \cline{2-5}
 & 4 & 8 & 4 & 9 & & & & & & & & &
\end{array}$$

4. **Determine the next candidate**: With $n' = 4849$ and $q = 1$, find the largest c_2 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 4849$

   - If $c_2 = 7$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1 \cdot 10^1) + 7 = 17$,

   $$f(7,1,17) = (10^1 \cdot 3 \cdot 7 \cdot 1 \cdot 17) + 7^3 = 3913 \leq 4849$$

   - If $c_2 = 8$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1 \cdot 10^1) + 8 = 18$,

   $$f(8,1,18) = (10^1 \cdot 3 \cdot 8 \cdot 1 \cdot 18) + 8^3 = 4832 \leq 4849$$

   - If $c_2 = 9$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1 \cdot 10^1) + 9 = 19$,

   $$f(9,1,19) = (10^1 \cdot 3 \cdot 9 \cdot 1 \cdot 19) + 9^3 = 5859 > 4849$$

$$\begin{array}{rccccccccccccc}
 & 1 & 8 & & & & & & & & & & & \\ \cline{2-14}
1) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
 & 1 & & & & & & & & & & & & \\ \cline{2-5}
18 & 4 & 8 & 4 & 9 & & & & & & & & & \\
 & 4 & 8 & 3 & 2 & & & & & & & & & \\ \cline{2-5}
 & & & 1 & 7 & & & & & & & & &
\end{array}$$

Choose $c_2 = 8$. Set $q = 18$, remainder $r = 4849 - 4832 = 17$.

5. **Bring down the next group**: Bring the next group 513 and append to $r = 17$, forming $n' = 17513$.

$$\begin{array}{rccccccccccccc}
 & 1 & 8 & & & & & & & & & & & \\ \cline{2-14}
1) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
 & 1 & & & & & & & & & & & & \\ \cline{2-5}
18 & 4 & 8 & 4 & 9 & & & & & & & & & \\
 & 4 & 8 & 3 & 2 & & & & & & & & & \\ \cline{2-8}
 & & & 1 & 7 & 5 & 1 & 3 & & & & & &
\end{array}$$

6. **Determine the next candidate**: With $n' = 17513$ and $q = 18$, find the largest c_3 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 17513$

   - If $c_3 = 0$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (18 \cdot 10^1) + 0 = 180$,

   $$f(0,18,180) = (10^1 \cdot 3 \cdot 0 \cdot 18 \cdot 180) + 0^3 = 0 \leq 17513$$

   - If $c_3 = 1$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (18 \cdot 10^1) + 1 = 181$,

   $$f(1,18,181) = (10^1 \cdot 3 \cdot 1 \cdot 18 \cdot 181) + 1^3 = 97741 > 17513$$

Choose $c_3 = 0$. Set $q = 180$, remainder $r = 17513 - 0 = 17513$.

$$
\begin{array}{rccccccccccccc}
 & 1 & 8 & 0 \\ \cline{2-14}
1) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
 & 1 \\ \cline{2-5}
18 & 4 & 8 & 4 & 9 \\
 & 4 & 8 & 3 & 2 \\ \cline{2-8}
180 & & & 1 & 7 & 5 & 1 & 3 \\
 & & & & & & & 0 \\ \cline{2-8}
 & & & 1 & 7 & 5 & 1 & 3
\end{array}
$$

7. **Bring down the next group**: Bring the next group 501 and append to $r = 17513$, forming $n' = 17513501$.

$$
\begin{array}{rccccccccccccc}
 & 1 & 8 & 0 \\ \cline{2-14}
1) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
 & 1 \\ \cline{2-5}
18 & 4 & 8 & 4 & 9 \\
 & 4 & 8 & 3 & 2 \\ \cline{2-8}
180 & & & 1 & 7 & 5 & 1 & 3 \\
 & & & & & & & 0 \\ \cline{2-11}
 & & & 1 & 7 & 5 & 1 & 3 & 5 & 0 & 1
\end{array}
$$

8. **Determine the next candidate**: With $n' = 17513501$ and $q = 180$, find the largest c_4 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 17513501$

   - If $c_4 = 0$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (180 \cdot 10^1) + 0 = 1800$,

     $$f(0, 180, 1800) = (10^1 \cdot 3 \cdot 0 \cdot 180 \cdot 1800) + 0^3 = 0 \leq 17513501$$

   - If $c_4 = 1$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (180 \cdot 10^1) + 1 = 1801$,

     $$f(1, 180, 1801) = (10^1 \cdot 3 \cdot 1 \cdot 180 \cdot 1801) + 1^3 = 9725401 \leq 17513501$$

   - If $c_4 = 2$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (180 \cdot 10^1) + 2 = 1802$,

     $$f(2, 180, 1802) = (10^1 \cdot 3 \cdot 2 \cdot 180 \cdot 1802) + 2^3 = 19461608 > 17513501$$

$$
\begin{array}{rccccccccccccc}
 & 1 & 8 & 0 & 1 \\ \cline{2-14}
1) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
 & 1 \\ \cline{2-5}
18 & 4 & 8 & 4 & 9 \\
 & 4 & 8 & 3 & 2 \\ \cline{2-8}
180 & & & 1 & 7 & 5 & 1 & 3 \\
 & & & & & & & 0 \\ \cline{2-11}
1801 & & & 1 & 7 & 5 & 1 & 3 & 5 & 0 & 1 \\
 & & & & 9 & 7 & 2 & 5 & 4 & 0 & 1 \\ \cline{2-11}
 & & & & 7 & 7 & 8 & 8 & 1 & 0 & 0
\end{array}
$$

Choose $c_4 = 1$. Set $q = 1801$, remainder $r = 17513501 - 9725401 = 7788100$.

9. **Bring down the next group**: Bring the next group 832 and append to $r = 7788100$, forming $n' = 7788100832$.

$$
\begin{array}{rccccccccccccc}
 & 1 & 8 & 0 & 1 \\ \cline{2-14}
1) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
 & 1 \\ \cline{2-5}
18 & 4 & 8 & 4 & 9 \\
 & 4 & 8 & 3 & 2 \\ \cline{2-8}
180 & & & 1 & 7 & 5 & 1 & 3 \\
 & & & & & & & 0 \\ \cline{2-11}
1801 & & & 1 & 7 & 5 & 1 & 3 & 5 & 0 & 1 \\
 & & & & 9 & 7 & 2 & 5 & 4 & 0 & 1 \\ \cline{2-14}
 & & & & 7 & 7 & 8 & 8 & 1 & 0 & 0 & 8 & 3 & 2
\end{array}
$$

10. **Determine the next candidate**: With $n' = 7788100832$ and $q = 1801$, find the largest c_5 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 7788100832$

- If $c_5 = 7$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1801 \cdot 10^1) + 7 = 18017$,
$$f(7, 1801, 18017) = (10^1 \cdot 3 \cdot 7 \cdot 1801 \cdot 18017) + 7^3 = 6814209913 \leq 7788100832$$
- If $c_5 = 8$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1801 \cdot 10^1) + 8 = 18018$,
$$f(8, 1801, 18018) = (10^1 \cdot 3 \cdot 8 \cdot 1801 \cdot 18018) + 8^3 = 7788100832 \leq 7788100832$$
- If $c_5 = 9$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1801 \cdot 10^1) + 9 = 18019$,
$$f(9, 1801, 18019) = (10^1 \cdot 3 \cdot 9 \cdot 1801 \cdot 18019) + 9^3 = 8762099859 > 7788100832$$

$$\begin{array}{r|ccccccccccccc}
 & 1 & 8 & 0 & 1 & 8 & & & & & & & & \\ \hline
1) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
 & 1 & & & & & & & & & & & & \\ \cline{2-5}
18 & 4 & 8 & 4 & 9 & & & & & & & & & \\
 & 4 & 8 & 3 & 2 & & & & & & & & & \\ \cline{2-8}
180 & & & 1 & 7 & 5 & 1 & 3 & & & & & & \\
 & & & & & & & 0 & & & & & & \\ \cline{2-11}
1801 & & & 1 & 7 & 5 & 1 & 3 & 5 & 0 & 1 & & & \\
 & & & & 9 & 7 & 2 & 5 & 4 & 0 & 1 & & & \\ \cline{2-14}
18018 & & & & 7 & 7 & 8 & 8 & 1 & 0 & 0 & 8 & 3 & 2 \\
 & & & & 7 & 7 & 8 & 8 & 1 & 0 & 0 & 8 & 3 & 2 \\ \cline{2-14}
 & & & & & & & & & & & & & 0 \\ \cline{2-14}
\end{array}$$

Choose $c_5 = 8$. Set $q = 18018$, remainder $r = 7788100832 - 7788100832 = 0$.

11. **Iterate or terminate**: No groups remaining, and $r = 0$, so the cube root is exact: $q = 18018$.
12. **Output**: Thus, $\sqrt[3]{5,849,513,501,832} = 18018$, computed by the long division method.

3.1.2 Flexible Grouping: $5849 \,|\, 513501832$

In this example,We partition the digits as follows, and group the partitions as below:

$$\underbrace{\underbrace{5}_{\text{Partition 1}} \;|\; \underbrace{849}_{\text{Partition 2}}}_{\text{Group 1}} \;|\; \underbrace{\underbrace{513}_{\text{Partition 3}} \;|\; \underbrace{501}_{\text{Partition 4}} \;|\; \underbrace{832}_{\text{Partition 5}}}_{\text{Group 2}}$$

1. **Group the digits**: Split 5,849,513,501,832 into partition by digit partition rule and group into computational units: $5849 \,|\, 513501832$.
2. **Initialize with the leftmost group**: With $n' = 5849$ and $q = 0$, find the largest c_1 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 5849$
   - If $c_1 = 17$, then $k = 2$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^2) + 17 = 17$,
   $$f(17, 0, 17) = (10^2 \cdot 3 \cdot 17 \cdot 0 \cdot 17) + 17^3 = 4913 \leq 5849$$
   - If $c_1 = 18$, then $k = 2$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^2) + 18 = 18$,
   $$f(18, 0, 18) = (10^2 \cdot 3 \cdot 18 \cdot 0 \cdot 18) + 18^3 = 5832 \leq 5849$$
   - If $c_1 = 19$, then $k = 2$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^2) + 19 = 19$,
   $$f(19, 0, 19) = (10^2 \cdot 3 \cdot 19 \cdot 0 \cdot 19) + 19^3 = 6859 > 5849$$

$$\begin{array}{r|ccccccccccccc}
 & 1 & 8 & & & & & & & & & & & \\ \hline
18) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
 & 5 & 8 & 3 & 2 & & & & & & & & & \\ \cline{2-5}
 & & & 1 & 7 & & & & & & & & & \\
\end{array}$$

Choose $c_1 = 18$. Set $q = 18$, remainder $r = 5849 - 5832 = 17$.

3. **Bring down the next group**: Bring the next group 513501832 and append to $r = 17$, forming $n' = 17513501832$.

$$\begin{array}{r|ccccccccccccc}
 & 1 & 8 & & & & & & & & & & & \\ \hline
18) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
 & 5 & 8 & 3 & 2 & & & & & & & & & \\ \cline{2-14}
 & & & 1 & 7 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
\end{array}$$

4. **Determine the next candidate**: With $n' = 17513501832$ and $q = 18$, find the largest c_2 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \le 17513501832$

   - If $c_2 = 017$, then $k = 3$ and $q' = (q \cdot 10^k) + c = (18 \cdot 10^3) + 017 = 18017$,

   $$f(017, 18, 18017) = (10^3 \cdot 3 \cdot 17 \cdot 18 \cdot 18017) + 17^3 = 16539610913 \le 17513501832$$

   - If $c_2 = 018$, then $k = 3$ and $q' = (q \cdot 10^k) + c = (18 \cdot 10^3) + 018 = 18018$,

   $$f(018, 18, 18018) = (10^3 \cdot 3 \cdot 18 \cdot 18 \cdot 18018) + 18^3 = 17513501832 \le 17513501832$$

   - If $c_2 = 019$, then $k = 3$ and $q' = (q \cdot 10^k) + c = (18 \cdot 10^3) + 019 = 18019$,

   $$f(019, 18, 18019) = (10^3 \cdot 3 \cdot 19 \cdot 18 \cdot 18019) + 19^3 = 18487500859 > 17513501832$$

$$\begin{array}{r|ccccccccccccc}
\multicolumn{1}{r}{} & 1 & 8 & 0 & 1 & 8 & & & & & & & & \\ \cline{2-14}
18) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
\multicolumn{1}{r}{} & 5 & 8 & 3 & 2 & & & & & & & & & \\ \cline{2-14}
\multicolumn{1}{r}{18018} & & & 1 & 7 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
\multicolumn{1}{r}{} & & & 1 & 7 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\ \cline{2-14}
\multicolumn{1}{r}{} & & & & & & & & & & & & & 0 \\ \cline{2-14}
\end{array}$$

   Choose $c_2 = 018$. Set $q = 18018$, remainder $r = 17513501832 - 17513501832 = 0$.

5. **Iterate or terminate**: No groups remaining, and $r = 0$, so the cube root is exact: $q = 18018$.

6. **Output**: Thus, $\sqrt[3]{5,849,513,501,832} = 18018$, computed by the long division method.

3.1.3 Flexible Grouping: $5 \,|\, 849513501 \,|\, 832$

In this example,We partition the digits as follows, and group the partitions as below:

$$\underbrace{\underbrace{5}_{\text{Partition 1}}}_{\text{Group 1}} \;\Big|\; \underbrace{\underbrace{849}_{\text{Partition 2}} \;\Big|\; \underbrace{513}_{\text{Partition 3}} \;\Big|\; \underbrace{501}_{\text{Partition 4}}}_{\text{Group 2}} \;\Big|\; \underbrace{\underbrace{832}_{\text{Partition 5}}}_{\text{Group 3}}$$

1. **Group the digits**: Split 5,849,513,501,832 into partition by digit partition rule and group into computational units: $5 \,|\, 849513501 \,|\, 832$.

2. **Initialize with the leftmost group**: With $n' = 5$ and $q = 0$, find the largest c_1 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \le 5$

   - If $c_1 = 0$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 0 = 0$,

   $$f(0, 0, 0) = (10^1 \cdot 3 \cdot 0 \cdot 0 \cdot 0) + 0^3 = 0 \le 5$$

   - If $c_1 = 1$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 1 = 1$,

   $$f(1, 0, 1) = (10^1 \cdot 3 \cdot 1 \cdot 0 \cdot 1) + 1^3 = 1 \le 5$$

   - If $c_1 = 2$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 2 = 2$,

   $$f(2, 0, 2) = (10^1 \cdot 3 \cdot 2 \cdot 0 \cdot 2) + 2^3 = 8 > 5$$

$$\begin{array}{r|ccccccccccccc}
\multicolumn{1}{r}{} & 1 & & & & & & & & & & & & \\ \cline{2-14}
1) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
\multicolumn{1}{r}{} & 1 & & & & & & & & & & & & \\ \cline{2-2}
\multicolumn{1}{r}{} & 4 & & & & & & & & & & & & \\
\end{array}$$

   Choose $c_1 = 1$. Set $q = 1$, remainder $r = 5 - 1 = 4$.

3. **Bring down the next group**: Bring the next group 849513501 and append to $r = 4$, forming $n' = 4849513501$.

$$\begin{array}{r|ccccccccccccc}
\multicolumn{1}{r}{} & 1 & & & & & & & & & & & & \\ \cline{2-14}
1) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
\multicolumn{1}{r}{} & 1 & & & & & & & & & & & & \\ \cline{2-11}
\multicolumn{1}{r}{} & 4 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & & & \\
\end{array}$$

4. **Determine the next candidate**: With $n' = 4849513501$ and $q = 1$, find the largest c_2 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 4849513501$

   - If $c_2 = 800$, then $k = 3$ and $q' = (q \cdot 10^k) + c = (1 \cdot 10^3) + 800 = 1800$,
   $$f(800, 1, 1800) = (10^3 \cdot 3 \cdot 800 \cdot 1 \cdot 1800) + 800^3 = 4832000000 \leq 4849513501$$
   - If $c_2 = 801$, then $k = 3$ and $q' = (q \cdot 10^k) + c = (1 \cdot 10^3) + 801 = 1801$,
   $$f(801, 1, 1801) = (10^3 \cdot 3 \cdot 801 \cdot 1 \cdot 1801) + 801^3 = 4841725401 \leq 4849513501$$
   - If $c_2 = 802$, then $k = 3$ and $q' = (q \cdot 10^k) + c = (1 \cdot 10^3) + 802 = 1802$,
   $$f(802, 1, 1802) = (10^3 \cdot 3 \cdot 802 \cdot 1 \cdot 1802) + 802^3 = 4851461608 > 4849513501$$

$$
\begin{array}{rccccccccccccc}
 & 1 & 8 & 0 & 1 & & & & & & & & & \\
\cline{2-14}
1) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
 & 1 & & & & & & & & & & & & \\
\cline{2-11}
1801 & 4 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & & & \\
 & 4 & 8 & 4 & 1 & 7 & 2 & 5 & 4 & 0 & 1 & & & \\
\cline{2-11}
 & & & & 7 & 7 & 8 & 8 & 1 & 0 & 0 & & & \\
\end{array}
$$

   Choose $c_2 = 801$. Set $q = 1801$, remainder $r = 4849513501 - 4841725401 = 7788100$.

5. **Bring down the next group**: Bring the next group 832 and append to $r = 7788100$, forming $n' = 7788100832$.

$$
\begin{array}{rccccccccccccc}
 & 1 & 8 & 0 & 1 & & & & & & & & & \\
\cline{2-14}
1) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
 & 1 & & & & & & & & & & & & \\
\cline{2-11}
1801 & 4 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & & & \\
 & 4 & 8 & 4 & 1 & 7 & 2 & 5 & 4 & 0 & 1 & & & \\
\cline{2-14}
 & & & & 7 & 7 & 8 & 8 & 1 & 0 & 0 & 8 & 3 & 2 \\
\end{array}
$$

6. **Determine the next candidate**: With $n' = 7788100832$ and $q = 1801$, find the largest c_3 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 7788100832$

   - If $c_3 = 7$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1801 \cdot 10^1) + 7 = 18017$,
   $$f(7, 1801, 18017) = (10^1 \cdot 3 \cdot 7 \cdot 1801 \cdot 18017) + 7^3 = 6814209913 \leq 7788100832$$
   - If $c_3 = 8$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1801 \cdot 10^1) + 8 = 18018$,
   $$f(8, 1801, 18018) = (10^1 \cdot 3 \cdot 8 \cdot 1801 \cdot 18018) + 8^3 = 7788100832 \leq 7788100832$$
   - If $c_3 = 9$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1801 \cdot 10^1) + 9 = 18019$,
   $$f(9, 1801, 18019) = (10^1 \cdot 3 \cdot 9 \cdot 1801 \cdot 18019) + 9^3 = 8762099859 > 7788100832$$

$$
\begin{array}{rccccccccccccc}
 & 1 & 8 & 0 & 1 & 8 & & & & & & & & \\
\cline{2-14}
1) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
 & 1 & & & & & & & & & & & & \\
\cline{2-11}
1801 & 4 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & & & \\
 & 4 & 8 & 4 & 1 & 7 & 2 & 5 & 4 & 0 & 1 & & & \\
\cline{2-14}
18018 & & & & 7 & 7 & 8 & 8 & 1 & 0 & 0 & 8 & 3 & 2 \\
 & & & & 7 & 7 & 8 & 8 & 1 & 0 & 0 & 8 & 3 & 2 \\
\cline{2-14}
 & & & & & & & & & & & & & 0 \\
\cline{2-14}
\end{array}
$$

   Choose $c_3 = 8$. Set $q = 18018$, remainder $r = 7788100832 - 7788100832 = 0$.

7. **Iterate or terminate**: No groups remaining, and $r = 0$, so the cube root is exact: $q = 18018$.

8. **Output**: Thus, $\sqrt[3]{5,849,513,501,832} = 18018$, computed by the long division method.

3.1.4 Flexible Grouping: $5849 \mid 513 \mid 501832$

In this example,We partition the digits as follows, and group the partitions as below:

$$\underbrace{\underbrace{5}_{\text{Partition 1}} \mid \underbrace{849}_{\text{Partition 2}}}_{\text{Group 1}} \mid \underbrace{\underbrace{513}_{\text{Partition 3}}}_{\text{Group 2}} \mid \underbrace{\underbrace{501}_{\text{Partition 4}} \mid \underbrace{832}_{\text{Partition 5}}}_{\text{Group 3}}$$

1. **Group the digits**: Split 5,849,513,501,832 into partition by digit partition rule and group into computational units: $5849 \mid 513 \mid 501832$.
2. **Initialize with the leftmost group**: With $n' = 5849$ and $q = 0$, find the largest c_1 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 5849$
   - If $c_1 = 17$, then $k = 2$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^2) + 17 = 17$,
   $$f(17, 0, 17) = (10^2 \cdot 3 \cdot 17 \cdot 0 \cdot 17) + 17^3 = 4913 \leq 5849$$
   - If $c_1 = 18$, then $k = 2$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^2) + 18 = 18$,
   $$f(18, 0, 18) = (10^2 \cdot 3 \cdot 18 \cdot 0 \cdot 18) + 18^3 = 5832 \leq 5849$$
   - If $c_1 = 19$, then $k = 2$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^2) + 19 = 19$,
   $$f(19, 0, 19) = (10^2 \cdot 3 \cdot 19 \cdot 0 \cdot 19) + 19^3 = 6859 > 5849$$

   $$\begin{array}{rccccccccccccc} & 1 & 8 \\ \cline{2-14} 18) & 5&8&4&9&5&1&3&5&0&1&8&3&2 \\ & 5&8&3&2 \\ \cline{2-5} & & &1&7 \end{array}$$

   Choose $c_1 = 18$. Set $q = 18$, remainder $r = 5849 - 5832 = 17$.
3. **Bring down the next group**: Bring the next group 513 and append to $r = 17$, forming $n' = 17513$.

   $$\begin{array}{rccccccccccccc} & 1 & 8 \\ \cline{2-14} 18) & 5&8&4&9&5&1&3&5&0&1&8&3&2 \\ & 5&8&3&2 \\ \cline{2-8} & & &1&7&5&1&3 \end{array}$$

4. **Determine the next candidate**: With $n' = 17513$ and $q = 18$, find the largest c_2 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 17513$
   - If $c_2 = 0$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (18 \cdot 10^1) + 0 = 180$,
   $$f(0, 18, 180) = (10^1 \cdot 3 \cdot 0 \cdot 18 \cdot 180) + 0^3 = 0 \leq 17513$$
   - If $c_2 = 1$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (18 \cdot 10^1) + 1 = 181$,
   $$f(1, 18, 181) = (10^1 \cdot 3 \cdot 1 \cdot 18 \cdot 181) + 1^3 = 97741 > 17513$$

   $$\begin{array}{rccccccccccccc} & 1 & 8 & 0 \\ \cline{2-14} 18) & 5&8&4&9&5&1&3&5&0&1&8&3&2 \\ & 5&8&3&2 \\ \cline{2-8} 180 & & &1&7&5&1&3 \\ & & & & & & &0 \\ \cline{2-8} & & &1&7&5&1&3 \end{array}$$

   Choose $c_2 = 0$. Set $q = 180$, remainder $r = 17513 - 0 = 17513$.
5. **Bring down the next group**: Bring the next group 501832 and append to $r = 17513$, forming $n' = 17513501832$.

   $$\begin{array}{rccccccccccccc} & 1 & 8 & 0 \\ \cline{2-14} 18) & 5&8&4&9&5&1&3&5&0&1&8&3&2 \\ & 5&8&3&2 \\ \cline{2-8} 180 & & &1&7&5&1&3 \\ & & & & & & &0 \\ \cline{2-14} & & &1&7&5&1&3&5&0&1&8&3&2 \end{array}$$

6. **Determine the next candidate**: With $n' = 17513501832$ and $q = 180$, find the largest c_3 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 17513501832$

   - If $c_3 = 17$, then $k = 2$ and $q' = (q \cdot 10^k) + c = (180 \cdot 10^2) + 17 = 18017$,
   $$f(17, 180, 18017) = (10^2 \cdot 3 \cdot 17 \cdot 180 \cdot 18017) + 17^3 = 16539610913 \leq 17513501832$$
   - If $c_3 = 18$, then $k = 2$ and $q' = (q \cdot 10^k) + c = (180 \cdot 10^2) + 18 = 18018$,
   $$f(18, 180, 18018) = (10^2 \cdot 3 \cdot 18 \cdot 180 \cdot 18018) + 18^3 = 17513501832 \leq 17513501832$$
   - If $c_3 = 19$, then $k = 2$ and $q' = (q \cdot 10^k) + c = (180 \cdot 10^2) + 19 = 18019$,
   $$f(19, 180, 18019) = (10^2 \cdot 3 \cdot 19 \cdot 180 \cdot 18019) + 19^3 = 18487500859 > 17513501832$$

$$
\begin{array}{r ccccccccccccc}
 & 1 & 8 & 0 & 1 & 8 \\ \hline
18) & 5 & 8 & 4 & 9 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
 & 5 & 8 & 3 & 2 \\ \cline{2-8}
180 & & & 1 & 7 & 5 & 1 & 3 \\
 & & & & & & & 0 \\ \hline
18018 & & & 1 & 7 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\
 & & & 1 & 7 & 5 & 1 & 3 & 5 & 0 & 1 & 8 & 3 & 2 \\ \hline
 & & & & & & & & & & & & & 0 \\ \hline
\end{array}
$$

   Choose $c_3 = 18$. Set $q = 18018$, remainder $r = 17513501832 - 17513501832 = 0$.

7. **Iterate or terminate**: No groups remaining, and $r = 0$, so the cube root is exact: $q = 18018$.
8. **Output**: Thus, $\sqrt[3]{5,849,513,501,832} = 18018$, computed by the long division method.

3.2 Non-perfect Cube Case

This subsection demonstrates the algorithm on a non-perfect cube where the cube root is not an integer. We use $\sqrt[3]{133387025} \approx 510.941$. The same number is computed multiple times with different digit groupings to show the result is unchanged by grouping. Extra appended zero partitions after the decimal (each partition 000) permit additional decimal precision; here we stop after three decimal digits.

3.2.1 Standard Grouping: 133 | 387 | 025 | .000 | 000 | 000

In this example, We are partitioning the digits as below, and we are planning to find till 3 decimal places, Hence added 3 partitions with decimal. The partitions are grouped as follows:

$$
\underbrace{\underbrace{133}_{\text{Partition 1}}}_{\text{Group 1}} \;\Big|\; \underbrace{\underbrace{387}_{\text{Partition 2}}}_{\text{Group 2}} \;\Big|\; \underbrace{\underbrace{025}_{\text{Partition 3}}}_{\text{Group 3}} \;\Big|\; \underbrace{\underbrace{.000}_{\text{Partition 4}}}_{\text{Group 4}} \;\Big|\; \underbrace{\underbrace{000}_{\text{Partition 5}}}_{\text{Group 5}} \;\Big|\; \underbrace{\underbrace{000}_{\text{Partition 5}}}_{\text{Group 6}}
$$

1. **Group the digits**: Split 133,387,025 into partition by digit partition rule and group into computational units: 133 | 387 | 025, We are planning to calculate up to 3 decimals, we can add 3 more partitions at the end.

   Hence: 133 | 387 | 025 | .000 | 000 | 000
2. **Initialize with the leftmost group**: With $n' = 133$ and $q = 0$, find the largest c_1 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 133$

   - If $c_1 = 4$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 4 = 4$,
   $$f(4, 0, 4) = (10^1 \cdot 3 \cdot 4 \cdot 0 \cdot 4) + 4^3 = 64 \leq 133$$
   - If $c_1 = 5$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 5 = 5$,
   $$f(5, 0, 5) = (10^1 \cdot 3 \cdot 5 \cdot 0 \cdot 5) + 5^3 = 125 \leq 133$$
   - If $c_1 = 6$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 6 = 6$,
   $$f(6, 0, 6) = (10^1 \cdot 3 \cdot 6 \cdot 0 \cdot 6) + 6^3 = 216 > 133$$

$$
\begin{array}{rcccccccccccccccccccc}
 & 5 \\ \cline{2-20}
5) & 1&3&3&3&8&7&0&2&5&\cdot&0&0&0&0&0&0&0&0&0\\
 & 1&2&5\\ \cline{2-4}
 & & &8
\end{array}
$$

Choose $c_1 = 5$. Set $q = 5$, remainder $r = 133 - 125 = 8$.

3. **Bring down the next group**: Bring the next group 387 and append to $r = 8$, forming $n' = 8387$.

$$
\begin{array}{rcccccccccccccccccccc}
 & 5 \\ \cline{2-20}
5) & 1&3&3&3&8&7&0&2&5&\cdot&0&0&0&0&0&0&0&0&0\\
 & 1&2&5\\ \cline{2-7}
 & & &8&3&8&7
\end{array}
$$

4. **Determine the next candidate**: With $n' = 8387$ and $q = 5$, find the largest c_2 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 8387$

   - If $c_2 = 0$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (5 \cdot 10^1) + 0 = 50$,
   $$f(0, 5, 50) = (10^1 \cdot 3 \cdot 0 \cdot 5 \cdot 50) + 0^3 = 0 \leq 8387$$
   - If $c_2 = 1$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (5 \cdot 10^1) + 1 = 51$,
   $$f(1, 5, 51) = (10^1 \cdot 3 \cdot 1 \cdot 5 \cdot 51) + 1^3 = 7651 \leq 8387$$
   - If $c_2 = 2$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (5 \cdot 10^1) + 2 = 52$,
   $$f(2, 5, 52) = (10^1 \cdot 3 \cdot 2 \cdot 5 \cdot 52) + 2^3 = 15608 > 8387$$

$$
\begin{array}{rcccccccccccccccccccc}
 & 5 & 1 \\ \cline{2-20}
5) & 1&3&3&3&8&7&0&2&5&\cdot&0&0&0&0&0&0&0&0&0\\
 & 1&2&5\\ \cline{2-7}
51 & & &8&3&8&7\\
 & & &7&6&5&1\\ \cline{2-7}
 & & & &7&3&6
\end{array}
$$

Choose $c_2 = 1$. Set $q = 51$, remainder $r = 8387 - 7651 = 736$.

5. **Bring down the next group**: Bring the next group 025 and append to $r = 736$, forming $n' = 736025$.

$$
\begin{array}{rcccccccccccccccccccc}
 & 5 & 1 \\ \cline{2-20}
5) & 1&3&3&3&8&7&0&2&5&\cdot&0&0&0&0&0&0&0&0&0\\
 & 1&2&5\\ \cline{2-7}
51 & & &8&3&8&7\\
 & & &7&6&5&1\\ \cline{2-10}
 & & & &7&3&6&0&2&5
\end{array}
$$

6. **Determine the next candidate**: With $n' = 736025$ and $q = 51$, find the largest c_3 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 736025$

   - If $c_3 = 0$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (51 \cdot 10^1) + 0 = 510$,
   $$f(0, 51, 510) = (10^1 \cdot 3 \cdot 0 \cdot 51 \cdot 510) + 0^3 = 0 \leq 736025$$
   - If $c_3 = 1$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (51 \cdot 10^1) + 1 = 511$,
   $$f(1, 51, 511) = (10^1 \cdot 3 \cdot 1 \cdot 51 \cdot 511) + 1^3 = 781831 > 736025$$

$$
\begin{array}{rcccccccccccccccccccc}
 & 5 & 1 & 0 \\ \cline{2-20}
5) & 1&3&3&3&8&7&0&2&5&\cdot&0&0&0&0&0&0&0&0&0\\
 & 1&2&5\\ \cline{2-7}
51 & & &8&3&8&7\\
 & & &7&6&5&1\\ \cline{2-10}
510 & & & &7&3&6&0&2&5\\
 & & & & & & & & &0\\ \cline{2-10}
 & & & &7&3&6&0&2&5
\end{array}
$$

Choose $c_3 = 0$. Set $q = 510$, remainder $r = 736025 - 0 = 736025$.

7. **Iterate or terminate**: No Groups are remaining, and $r = 736025 \neq 0$, Hence it is not a perfect cube, For precision we can add more partitions after decimal and bring them either one by one, or multiple partitions at same time. Bring the next group .000 and append to $r = 736025$, forming $n' = 736025.000$.

$$
\begin{array}{r|ccccccccccccccccccc}
 & 5 & 1 & 0 \\
\hline
5) & 1 & 3 & 3 & 3 & 8 & 7 & 0 & 2 & 5 & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 2 & 5 \\
51 & & & 8 & 3 & 8 & 7 \\
 & & & 7 & 6 & 5 & 1 \\
510 & & & & 7 & 3 & 6 & 0 & 2 & 5 \\
 & & & & & & & & & 0 \\
 & & & & 7 & 3 & 6 & 0 & 2 & 5 & \cdot & 0 & 0 & 0
\end{array}
$$

8. **Determine the next candidate**: With $n' = 736025.000$ and $q = 510$, find the largest c_4 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 736025.000$

   - If $c_4 = .8$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (510 \cdot 10^0) + .8 = 510.8$,

   $$f(.8, 510, 510.8) = (10^0 \cdot 3 \cdot 0.8 \cdot 510 \cdot 510.8) + 0.8^3 = 625219.712 \leq 736025.000$$

   - If $c_4 = .9$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (510 \cdot 10^0) + .9 = 510.9$,

   $$f(.9, 510, 510.9) = (10^0 \cdot 3 \cdot 0.9 \cdot 510 \cdot 510.9) + 0.9^3 = 703510.029 \leq 736025.000$$

   - If $c_4 = 1.0$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (510 \cdot 10^1) + 1.0 = 5101$,

   $$f(1.0, 510, 5101) = (10^1 \cdot 3 \cdot 1 \cdot 510 \cdot 5101) + 1^3 = 78045301 > 736025.000$$

$$
\begin{array}{r|ccccccccccccccccccc}
 & 5 & 1 & 0 & \cdot & 9 \\
\hline
5) & 1 & 3 & 3 & 3 & 8 & 7 & 0 & 2 & 5 & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 2 & 5 \\
51 & & & 8 & 3 & 8 & 7 \\
 & & & 7 & 6 & 5 & 1 \\
510 & & & & 7 & 3 & 6 & 0 & 2 & 5 \\
 & & & & & & & & & 0 \\
510.9 & & & & 7 & 3 & 6 & 0 & 2 & 5 & \cdot & 0 & 0 & 0 \\
 & & & & 7 & 0 & 3 & 5 & 1 & 0 & \cdot & 0 & 2 & 9 \\
 & & & & & 3 & 2 & 5 & 1 & 4 & \cdot & 9 & 7 & 1
\end{array}
$$

Choose $c_4 = .9$. Set $q = 510.9$, remainder $r = 736025.000 - 703510.029 = 32514.971$.

9. **Iterate or terminate**: No Groups in integer part are remaining, and $r = 32514.971 \neq 0$, For precision we can add more partitions after decimal and bring them either one by one, or multiple partitions at same time. Bring the next group 000 and append to $r = 32514.971$, forming $n' = 32514.971000$.

$$
\begin{array}{r|ccccccccccccccccccc}
 & 5 & 1 & 0 & \cdot & 9 \\
\hline
5) & 1 & 3 & 3 & 3 & 8 & 7 & 0 & 2 & 5 & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 2 & 5 \\
51 & & & 8 & 3 & 8 & 7 \\
 & & & 7 & 6 & 5 & 1 \\
510 & & & & 7 & 3 & 6 & 0 & 2 & 5 \\
 & & & & & & & & & 0 \\
510.9 & & & & 7 & 3 & 6 & 0 & 2 & 5 & \cdot & 0 & 0 & 0 \\
 & & & & 7 & 0 & 3 & 5 & 1 & 0 & \cdot & 0 & 2 & 9 \\
 & & & & & 3 & 2 & 5 & 1 & 4 & \cdot & 9 & 7 & 1 & 0 & 0 & 0
\end{array}
$$

10. **Determine the next candidate**: With $n' = 32514.971000$ and $q = 510.9$, find the largest c_5 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 32514.971000$

   - If $c_5 = .03$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (510.9 \cdot 10^0) + .03 = 510.93$,

   $$f(.03, 510.9, 510.93) = (10^0 \cdot 3 \cdot 0.03 \cdot 510.9 \cdot 510.93) + 0.03^3 = 23493.072357 \leq 32514.971000$$

   - If $c_5 = .04$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (510.9 \cdot 10^0) + .04 = 510.94$,

   $$f(.04, 510.9, 510.94) = (10^0 \cdot 3 \cdot 0.04 \cdot 510.9 \cdot 510.94) + 0.04^3 = 31324.709584 \leq 32514.971000$$

- If $c_5 = .05$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (510.9 \cdot 10^0) + .05 = 510.95$,

$$f(.05, 510.9, 510.95) = (10^0 \cdot 3 \cdot 0.05 \cdot 510.9 \cdot 510.95) + 0.05^3 = 39156.653375 > 32514.971000$$

	5	1	0	·	9	4													
5)	1	3	3	3	8	7	0	2	5	·	0	0	0	0	0	0	0	0	0
	1	2	5																
51			8	3	8	7													
			7	6	5	1													
510				7	3	6	0	2	5										
									0										
510.9				7	3	6	0	2	5	·	0	0	0						
				7	0	3	5	1	0	·	0	2	9						
510.94					3	2	5	1	4	·	9	7	1	0	0	0			
					3	1	3	2	4	·	7	0	9	5	8	4			
						1	1	9	0	·	2	6	1	4	1	6			

Choose $c_5 = .04$. Set $q = 510.94$, remainder $r = 32514.971000 - 31324.709584 = 1190.261416$.

11. **Iterate or terminate**: No Groups in integer part are remaining, and $r = 1190.261416 \neq 0$, For precision we can add more partitions after decimal and bring them either one by one, or multiple partitions at same time. Bring the next group 000 and append to $r = 1190.261416$, forming $n' = 1190.261416000$.

	5	1	0	·	9	4													
5)	1	3	3	3	8	7	0	2	5	·	0	0	0	0	0	0	0	0	0
	1	2	5																
51			8	3	8	7													
			7	6	5	1													
510				7	3	6	0	2	5										
									0										
510.9				7	3	6	0	2	5	·	0	0	0						
				7	0	3	5	1	0	·	0	2	9						
510.94					3	2	5	1	4	·	9	7	1	0	0	0			
					3	1	3	2	4	·	7	0	9	5	8	4			
						1	1	9	0	·	2	6	1	4	1	6	0	0	0

12. **Determine the next candidate**: With $n' = 1190.261416000$ and $q = 510.94$, find the largest c_6 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 1190.261416000$

- If $c_6 = .000$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (510.94 \cdot 10^0) + .000 = 510.94$,

$$f(.000, 510.94, 510.94) = (10^0 \cdot 3 \cdot 0 \cdot 510.94 \cdot 510.94) + 0^3 = 0 \leq 1190.261416000$$

- If $c_6 = .001$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (510.94 \cdot 10^0) + .001 = 510.941$,

$$f(.001, 510.94, 510.941) = (10^0 \cdot 3 \cdot 0.001 \cdot 510.94 \cdot 510.941) + 0.001^3 = 783.180583621 \leq 1190.261416000$$

- If $c_6 = .002$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (510.94 \cdot 10^0) + .002 = 510.942$,

$$f(.002, 510.94, 510.942) = (10^0 \cdot 3 \cdot 0.002 \cdot 510.94 \cdot 510.942) + 0.002^3 = 1566.364232888 > 1190.261416000$$

	5	1	0	·	9	4	1												
5)	1	3	3	3	8	7	0	2	5	·	0	0	0	0	0	0	0	0	0
	1	2	5																
51			8	3	8	7													
			7	6	5	1													
510				7	3	6	0	2	5										
									0										
510.9				7	3	6	0	2	5	·	0	0	0						
				7	0	3	5	1	0	·	0	2	9						
510.94					3	2	5	1	4	·	9	7	1	0	0	0			
					3	1	3	2	4	·	7	0	9	5	8	4			
510.941						1	1	9	0	·	2	6	1	4	1	6	0	0	0
							7	8	3	·	1	8	0	5	8	3	6	2	1
							4	0	7	.	0	8	0	8	3	2	3	7	9

Choose $c_6 = .001$. Set $q = 510.941$, remainder $r = 1190.261416000 - 783.180583621 = 407.080832379$.

13. **Iterate or terminate**: No Groups in integer part are remaining, and $r = 407.080832379 \neq 0$, If we want we can bring more partitions for more precision, But will stop here with 3 decimals for this example.

14. **Output**: Thus, $\sqrt[3]{133,387,025} \approx 510.941$, computed by the long division method.

3.2.2 Flexible Grouping: 133387025 | .000000 | 000

In this example, We are partitioning the digits as below, and we are planning to find till 3 decimal places, Hence added 3 partitions with decimal. The partitions are grouped as follows:

$$\underbrace{\underbrace{133}_{\text{Partition 1}} \;|\; \underbrace{387}_{\text{Partition 2}} \;|\; \underbrace{025}_{\text{Partition 3}}}_{\text{Group 1}} \;|\; \underbrace{\underbrace{.000}_{\text{Partition 4}} \;|\; \underbrace{000}_{\text{Partition 5}}}_{\text{Group 2}} \;|\; \underbrace{\underbrace{000}_{\text{Partition 5}}}_{\text{Group 3}}$$

1. **Group the digits**: Split 133,387,025 into partition by digit partition rule and group into computational units and since we are planning to calculate till 3 decimals, added 3 partitions at the end with decimals: 133387025 | .000000 | 000

2. **Initialize with the leftmost group**: With $n' = 133387025$ and $q = 0$, find the largest c_1 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 133387025$

   - If $c_1 = 509$, then $k = 3$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^3) + 509 = 509$,

   $$f(509, 0, 509) = (10^3 \cdot 3 \cdot 509 \cdot 0 \cdot 509) + 509^3 = 131872229 \leq 133387025$$

   - If $c_1 = 510$, then $k = 3$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^3) + 510 = 510$,

   $$f(510, 0, 510) = (10^3 \cdot 3 \cdot 510 \cdot 0 \cdot 510) + 510^3 = 132651000 \leq 133387025$$

   - If $c_1 = 511$, then $k = 3$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^3) + 511 = 511$,

   $$f(511, 0, 511) = (10^3 \cdot 3 \cdot 511 \cdot 0 \cdot 511) + 511^3 = 133432831 > 133387025$$

$$\begin{array}{r|ccccccccccccccccccc}
 & 5 & 1 & 0 & & & & & & & & & & & & & & & & \\ \hline
510) & 1 & 3 & 3 & 3 & 8 & 7 & 0 & 2 & 5 & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 3 & 2 & 6 & 5 & 1 & 0 & 0 & 0 & & & & & & & & & & \\ \hline
 & & & & 7 & 3 & 6 & 0 & 2 & 5 & & & & & & & & & &
\end{array}$$

Choose $c_1 = 510$. Set $q = 510$, remainder $r = 133387025 - 132651000 = 736025$.

3. **Iterate or terminate**: No Groups in integer part are remaining, and $r = 736025 \neq 0$, For precision we can add more partitions after decimal and bring them either one by one, or multiple partitions at same time. Bring the next group which contains two partitions .000000 and append to $r = 736025$, forming $n' = 736025.000000$.

$$\begin{array}{r|ccccccccccccccccccc}
 & 5 & 1 & 0 & & & & & & & & & & & & & & & & \\ \hline
510) & 1 & 3 & 3 & 3 & 8 & 7 & 0 & 2 & 5 & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 3 & 2 & 6 & 5 & 1 & 0 & 0 & 0 & & & & & & & & & & \\ \hline
 & & & & 7 & 3 & 6 & 0 & 2 & 5 & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & & &
\end{array}$$

4. **Determine the next candidate**: With $n' = 736025.000000$ and $q = 510$, find the largest c_2 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 736025.000000$

   - If $c_2 = .93$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (510 \cdot 10^0) + .93 = 510.93$,

   $$f(.93, 510, 510.93) = (10^0 \cdot 3 \cdot 0.93 \cdot 510 \cdot 510.93) + 0.93^3 = 727003.101357 \leq 736025.000000$$

   - If $c_2 = .94$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (510 \cdot 10^0) + .94 = 510.94$,

   $$f(.94, 510, 510.94) = (10^0 \cdot 3 \cdot 0.94 \cdot 510 \cdot 510.94) + 0.94^3 = 734834.738584 \leq 736025.000000$$

   - If $c_2 = .95$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (510 \cdot 10^0) + .95 = 510.95$,

   $$f(.95, 510, 510.95) = (10^0 \cdot 3 \cdot 0.95 \cdot 510 \cdot 510.95) + 0.95^3 = 742666.682375 > 736025.000000$$

$$\begin{array}{r|ccccccccccccccccccc}
 & 5 & 1 & 0 & \cdot & 9 & 4 & & & & & & & & & & & & & \\ \hline
510) & 1 & 3 & 3 & 3 & 8 & 7 & 0 & 2 & 5 & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 3 & 2 & 6 & 5 & 1 & 0 & 0 & 0 & & & & & & & & & & \\ \hline
510.94 & & & & 7 & 3 & 6 & 0 & 2 & 5 & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
 & & & & 7 & 3 & 4 & 8 & 3 & 4 & \cdot & 7 & 3 & 8 & 5 & 8 & 4 & & & \\ \hline
 & & & & & & 1 & 1 & 9 & 0 & \cdot & 2 & 6 & 1 & 4 & 1 & 6 & & &
\end{array}$$

Choose $c_2 = .94$. Set $q = 510.94$, remainder $r = 736025.000000 - 734834.738584 = 1190.261416$.

5. **Bring down the next group**: Bring the next group 000 and append to $r = 1190.261416$, forming $n' = 1190.261416000$.

$$
\begin{array}{r|ccccccccccccccccccc}
 & 5 & 1 & 0 & \cdot & 9 & 4 & & & & & & & & & & & & & \\
\hline
510) & 1 & 3 & 3 & 3 & 8 & 7 & 0 & 2 & 5 & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 3 & 2 & 6 & 5 & 1 & 0 & 0 & 0 & & & & & & & & & & \\
\hline
510.94 & & & & 7 & 3 & 6 & 0 & 2 & 5 & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
 & & & & 7 & 3 & 4 & 8 & 3 & 4 & \cdot & 7 & 3 & 8 & 5 & 8 & 4 & & & \\
\hline
 & & & & & & 1 & 1 & 9 & 0 & \cdot & 2 & 6 & 1 & 4 & 1 & 6 & 0 & 0 & 0
\end{array}
$$

6. **Determine the next candidate**: With $n' = 1190.261416000$ and $q = 510.94$, find the largest c_3 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 1190.261416000$

   - If $c_3 = .000$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (510.94 \cdot 10^0) + .000 = 510.94$,

$$f(.000, 510.94, 510.94) = (10^0 \cdot 3 \cdot 0 \cdot 510.94 \cdot 510.94) + 0^3 = 0 \leq 1190.261416000$$

   - If $c_3 = .001$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (510.94 \cdot 10^0) + .001 = 510.941$,

$$f(.001, 510.94, 510.941) = (10^0 \cdot 3 \cdot 0.001 \cdot 510.94 \cdot 510.941) + 0.001^3 = 783.180583621 \leq 1190.261416000$$

   - If $c_3 = .002$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (510.94 \cdot 10^0) + .002 = 510.942$,

$$f(.002, 510.94, 510.942) = (10^0 \cdot 3 \cdot 0.002 \cdot 510.94 \cdot 510.942) + 0.002^3 = 1566.364232888 > 1190.261416000$$

$$
\begin{array}{r|ccccccccccccccccccc}
 & 5 & 1 & 0 & \cdot & 9 & 4 & 1 & & & & & & & & & & & & \\
\hline
510) & 1 & 3 & 3 & 3 & 8 & 7 & 0 & 2 & 5 & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 3 & 2 & 6 & 5 & 1 & 0 & 0 & 0 & & & & & & & & & & \\
\hline
510.94 & & & & 7 & 3 & 6 & 0 & 2 & 5 & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
 & & & & 7 & 3 & 4 & 8 & 3 & 4 & \cdot & 7 & 3 & 8 & 5 & 8 & 4 & & & \\
\hline
510.941 & & & & & & 1 & 1 & 9 & 0 & \cdot & 2 & 6 & 1 & 4 & 1 & 6 & 0 & 0 & 0 \\
 & & & & & & & 7 & 8 & 3 & \cdot & 1 & 8 & 0 & 5 & 8 & 3 & 6 & 2 & 1 \\
\hline
 & & & & & & & 4 & 0 & 7 & \cdot & 0 & 8 & 0 & 8 & 3 & 2 & 3 & 7 & 9
\end{array}
$$

   Choose $c_3 = .001$. Set $q = 510.941$, remainder $r = 1190.261416000 - 783.180583621 = 407.080832379$.

7. **Iterate or terminate**: No Groups in integer part are remaining, and $r = 407.080832379 \neq 0$, If we want we can bring more group for more precision, But will stop here with 3 decimals for this example.

8. **Output**: Thus, $\sqrt[3]{133,387,025} \approx 510.941$, computed by the long division method.

3.3 Decimal Radicand Case

This subsection demonstrates the algorithm on a Decimal Radicand Case where the number is a decimal. We use $\sqrt[3]{1218.2161} \approx 10.68$. The same number is computed multiple times with different digit groupings to show the result is unchanged by grouping. Extra appended zero partitions after the decimal (each partition 000) permit additional decimal precision

3.3.1 Standard Grouping: $1 \,|\, 218 \,|\, .216 \,|\, 1$

In this example,We partition the digits as follows, and group the partitions as below:

$$\underbrace{\underbrace{1}_{\text{Partition 1}}}_{\text{Group 1}} \;\Big|\; \underbrace{\underbrace{218}_{\text{Partition 2}}}_{\text{Group 2}} \;\Big|\; \underbrace{\underbrace{.216}_{\text{Partition 3}}}_{\text{Group 3}} \;\Big|\; \underbrace{\underbrace{100}_{\text{Partition 4}}}_{\text{Group 4}}$$

1. **Group the digits**: Split 1218.2161 into partition by digit partition rule and group into computational units: $1 \,|\, 218 \,|\, .216 \,|\, 1$ (or $001 \,|\, 218 \,|\, .216 \,|\, 100$, where the leftmost & rightmost group is padded as needed).

2. **Initialize with the leftmost group**: With $n' = 1$ and $q = 0$, find the largest c_1 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 1$

   - If $c_1 = 0$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 0 = 0$,

$$f(0, 0, 0) = (10^1 \cdot 3 \cdot 0 \cdot 0 \cdot 0) + 0^3 = 0 \leq 1$$

- If $c_1 = 1$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 1 = 1$,

$$f(1, 0, 1) = (10^1 \cdot 3 \cdot 1 \cdot 0 \cdot 1) + 1^3 = 1 \leq 1$$

- If $c_1 = 2$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 2 = 2$,

$$f(2, 0, 2) = (10^1 \cdot 3 \cdot 2 \cdot 0 \cdot 2) + 2^3 = 8 > 1$$

$$
\begin{array}{r|ccccccccccc}
 & 1 \\ \cline{2-12}
1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
 & 1 \\ \cline{2-2}
 & 0
\end{array}
$$

Choose $c_1 = 1$. Set $q = 1$, remainder $r = 1 - 1 = 0$.

3. **Bring down the next group**: Bring the next group 218 and append to $r = 0$, forming $n' = 0218$.

$$
\begin{array}{r|ccccccccccc}
 & 1 \\ \cline{2-12}
1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
 & 1 \\ \cline{2-5}
 & 0 & 2 & 1 & 8
\end{array}
$$

4. **Determine the next candidate**: With $n' = 0218$ and $q = 1$, find the largest c_2 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 0218$

- If $c_2 = 0$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1 \cdot 10^1) + 0 = 10$,

$$f(0, 1, 10) = (10^1 \cdot 3 \cdot 0 \cdot 1 \cdot 10) + 0^3 = 0 \leq 0218$$

- If $c_2 = 1$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1 \cdot 10^1) + 1 = 11$,

$$f(1, 1, 11) = (10^1 \cdot 3 \cdot 1 \cdot 1 \cdot 11) + 1^3 = 331 > 0218$$

$$
\begin{array}{r|ccccccccccc}
 & 1 & 0 \\ \cline{2-12}
1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
 & 1 \\ \cline{2-5}
10 & 0 & 2 & 1 & 8 \\
 & & & & 0 \\ \cline{2-5}
 & & 2 & 1 & 8
\end{array}
$$

Choose $c_2 = 0$. Set $q = 10$, remainder $r = 0218 - 0 = 218$.

5. **Bring down the next group**: Bring the next group .216 and append to $r = 218$, forming $n' = 218.216$.

$$
\begin{array}{r|ccccccccccc}
 & 1 & 0 \\ \cline{2-12}
1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
 & 1 \\ \cline{2-5}
10 & 0 & 2 & 1 & 8 \\
 & & & & 0 \\ \cline{2-9}
 & & 2 & 1 & 8 & \cdot & 2 & 1 & 6
\end{array}
$$

6. **Determine the next candidate**: With $n' = 218.216$ and $q = 10$, find the largest c_3 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 218.216$

- If $c_3 = .5$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (10 \cdot 10^0) + .5 = 10.5$,

$$f(.5, 10, 10.5) = (10^0 \cdot 3 \cdot 0.5 \cdot 10 \cdot 10.5) + 0.5^3 = 157.625 \leq 218.216$$

- If $c_3 = .6$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (10 \cdot 10^0) + .6 = 10.6$,

$$f(.6, 10, 10.6) = (10^0 \cdot 3 \cdot 0.6 \cdot 10 \cdot 10.6) + 0.6^3 = 191.016 \leq 218.216$$

- If $c_3 = .7$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (10 \cdot 10^0) + .7 = 10.7$,

$$f(.7, 10, 10.7) = (10^0 \cdot 3 \cdot 0.7 \cdot 10 \cdot 10.7) + 0.7^3 = 225.043 > 218.216$$

$$
\begin{array}{r|ccccccccccc}
 & 1 & 0 & \cdot & 6 & & & & & & & \\ \cline{2-12}
1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
 & 1 & & & & & & & & & & \\ \cline{2-5}
10 & 0 & 2 & 1 & 8 & & & & & & & \\
 & & & & 0 & & & & & & & \\ \cline{2-8}
10.6 & & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & & & \\
 & & 1 & 9 & 1 & \cdot & 0 & 1 & 6 & & & \\ \cline{2-8}
 & & & 2 & 7 & \cdot & 2 & 0 & 0 & & &
\end{array}
$$

Choose $c_3 = .6$. Set $q = 10.6$, remainder $r = 218.216 - 191.016 = 27.200$.

7. **Bring down the next group**: Bring the next group 100 and append to $r = 27.200$, forming $n' = 27.200100$.

$$
\begin{array}{r|ccccccccccc}
 & 1 & 0 & \cdot & 6 & & & & & & & \\ \cline{2-12}
1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
 & 1 & & & & & & & & & & \\ \cline{2-5}
10 & 0 & 2 & 1 & 8 & & & & & & & \\
 & & & & 0 & & & & & & & \\ \cline{2-8}
10.6 & & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & & & \\
 & & 1 & 9 & 1 & \cdot & 0 & 1 & 6 & & & \\ \cline{2-12}
 & & & 2 & 7 & \cdot & 2 & 0 & 0 & 1 & 0 & 0
\end{array}
$$

8. **Determine the next candidate**: With $n' = 27.200100$ and $q = 10.6$, find the largest c_4 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 27.200100$

   - If $c_4 = .07$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (10.6 \cdot 10^0) + .07 = 10.67$,

$$f(.07, 10.6, 10.67) = (10^0 \cdot 3 \cdot 0.07 \cdot 10.6 \cdot 10.67) + 0.07^3 = 23.751763 \leq 27.200100$$

   - If $c_4 = .08$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (10.6 \cdot 10^0) + .08 = 10.68$,

$$f(.08, 10.6, 10.68) = (10^0 \cdot 3 \cdot 0.08 \cdot 10.6 \cdot 10.68) + 0.08^3 = 27.170432 \leq 27.200100$$

   - If $c_4 = .09$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (10.6 \cdot 10^0) + .09 = 10.69$,

$$f(.09, 10.6, 10.69) = (10^0 \cdot 3 \cdot 0.09 \cdot 10.6 \cdot 10.69) + 0.09^3 = 30.595509 > 27.200100$$

$$
\begin{array}{r|ccccccccccc}
 & 1 & 0 & \cdot & 6 & 8 & & & & & & \\ \cline{2-12}
1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
 & 1 & & & & & & & & & & \\ \cline{2-5}
10 & 0 & 2 & 1 & 8 & & & & & & & \\
 & & & & 0 & & & & & & & \\ \cline{2-8}
10.6 & & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & & & \\
 & & 1 & 9 & 1 & \cdot & 0 & 1 & 6 & & & \\ \cline{2-12}
10.68 & & & 2 & 7 & \cdot & 2 & 0 & 0 & 1 & 0 & 0 \\
 & & & 2 & 7 & \cdot & 1 & 7 & 0 & 4 & 3 & 2 \\ \cline{2-12}
 & & & & 0 & \cdot & 0 & 2 & 9 & 6 & 6 & 8
\end{array}
$$

Choose $c_4 = .08$. Set $q = 10.68$, remainder $r = 27.200100 - 27.170432 = 0.029668$.

9. **Iterate or terminate**: No Groups remaining, and $r = 0.029668 \neq 0$, We can bring one more group 000 and append to $r = 0.029668$ to calculate one more decimal, forming $n' = 0.029668000$.

$$
\begin{array}{r|cccccccccccccc}
 & 1 & 0 & \cdot & 6 & 8 & & & & & & & & & \\ \cline{2-15}
1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & & & & & & & & & & & & & \\ \cline{2-5}
10 & 0 & 2 & 1 & 8 & & & & & & & & & & \\
 & & & & 0 & & & & & & & & & & \\ \cline{2-8}
10.6 & & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & & & & & & \\
 & & 1 & 9 & 1 & \cdot & 0 & 1 & 6 & & & & & & \\ \cline{2-12}
10.68 & & & 2 & 7 & \cdot & 2 & 0 & 0 & 1 & 0 & 0 & & & \\
 & & & 2 & 7 & \cdot & 1 & 7 & 0 & 4 & 3 & 2 & & & \\ \cline{2-15}
 & & & & 0 & \cdot & 0 & 2 & 9 & 6 & 6 & 8 & 0 & 0 & 0
\end{array}
$$

10. **Determine the next candidate**: With $n' = 0.029668000$ and $q = 10.68$, find the largest c_5 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 0.029668000$

- If $c_5 = .000$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (10.68 \cdot 10^0) + .000 = 10.68$,

$$f(.000, 10.68, 10.68) = (10^0 \cdot 3 \cdot 0 \cdot 10.68 \cdot 10.68) + 0^3 = 0 \leq 0.029668000$$

- If $c_5 = .001$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (10.68 \cdot 10^0) + .001 = 10.681$,

$$f(.001, 10.68, 10.681) = (10^0 \cdot 3 \cdot 0.001 \cdot 10.68 \cdot 10.681) + 0.001^3 = 0.342219241 > 0.029668000$$

$$\begin{array}{r|cccccccccccccc}
 & 1 & 0 & \cdot & 6 & 8 & 0 & & & & & & & & \\
\hline
1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & & & & & & & & & & & & & \\
\hline
10 & 0 & 2 & 1 & 8 & & & & & & & & & & \\
 & & & & 0 & & & & & & & & & & \\
\hline
10.6 & & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & & & & & & \\
 & & 1 & 9 & 1 & \cdot & 0 & 1 & 6 & & & & & & \\
\hline
10.68 & & & 2 & 7 & \cdot & 2 & 0 & 0 & 1 & 0 & 0 & & & \\
 & & & 2 & 7 & \cdot & 1 & 7 & 0 & 4 & 3 & 2 & & & \\
\hline
10.680 & & & & 0 & \cdot & 0 & 2 & 9 & 6 & 6 & 8 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & 0 \\
\hline
 & & & & 0 & \cdot & 0 & 2 & 9 & 6 & 6 & 8 & 0 & 0 & 0 \\
\hline
\end{array}$$

Choose $c_5 = .000$. Set $q = 10.680$, remainder $r = 0.029668000 - 0 = 0.029668000$.

11. **Iterate or terminate**: No groups remaining and $r = 0.029668000 \neq 0$. We can bring more group to find the more precisions, But will stop here with 3 decimals.

12. **Output**: Thus, $\sqrt[3]{1218.2161} \approx 10.680$, computed by the long division method.

3.3.2 Flexible Grouping: $1 \,|\, 218.216 \,|\, 1$

In this example,We partition the digits as follows, and group the partitions as below:

$$\underbrace{\underbrace{1}_{\text{Partition 1}}}_{\text{Group 1}} \;\Big|\; \underbrace{\underbrace{218}_{\text{Partition 2}} \;\Big|\; \underbrace{.216}_{\text{Partition 3}}}_{\text{Group 2}} \;\Big|\; \underbrace{\underbrace{100}_{\text{Partition 4}}}_{\text{Group 3}}$$

1. **Group the digits**: Split 1218.2161 into partition by digit partition rule and group into computational units: $1 \,|\, 218.216 \,|\, 1$ (or $001 \,|\, 218.216 \,|\, 100$, where the leftmost & rightmost group is padded as needed).

2. **Initialize with the leftmost group**: With $n' = 1$ and $q = 0$, find the largest c_1 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 1$

   - If $c_1 = 0$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 0 = 0$,

   $$f(0, 0, 0) = (10^1 \cdot 3 \cdot 0 \cdot 0 \cdot 0) + 0^3 = 0 \leq 1$$

   - If $c_1 = 1$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 1 = 1$,

   $$f(1, 0, 1) = (10^1 \cdot 3 \cdot 1 \cdot 0 \cdot 1) + 1^3 = 1 \leq 1$$

   - If $c_1 = 2$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 2 = 2$,

   $$f(2, 0, 2) = (10^1 \cdot 3 \cdot 2 \cdot 0 \cdot 2) + 2^3 = 8 > 1$$

   $$\begin{array}{r|cccccccccc}
    & 1 & & & & & & & & & \\
   \hline
   1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
    & 1 & & & & & & & & & \\
   \hline
    & 0 & & & & & & & & & \\
   \end{array}$$

   Choose $c_1 = 1$. Set $q = 1$, remainder $r = 1 - 1 = 0$.

3. **Bring down the next group**: Bring the next group 218.216 and append to $r = 0$, forming $n' = 218.216$.

   $$\begin{array}{r|cccccccccc}
    & 1 & & & & & & & & & \\
   \hline
   1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
    & 1 & & & & & & & & & \\
   \hline
    & 0 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & & & \\
   \end{array}$$

4. **Determine the next candidate**: With $n' = 218.216$ and $q = 1$, find the largest c_2 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 218.216$

   - If $c_2 = 0.5$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1 \cdot 10^1) + 0.5 = 10.5$,
   $$f(0.5, 1, 10.5) = (10^1 \cdot 3 \cdot 0.5 \cdot 1 \cdot 10.5) + 0.5^3 = 157.625 \leq 218.216$$
   - If $c_2 = 0.6$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1 \cdot 10^1) + 0.6 = 10.6$,
   $$f(0.6, 1, 10.6) = (10^1 \cdot 3 \cdot 0.6 \cdot 1 \cdot 10.6) + 0.6^3 = 191.016 \leq 218.216$$
   - If $c_2 = 0.7$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1 \cdot 10^1) + 0.7 = 10.7$,
   $$f(0.7, 1, 10.7) = (10^1 \cdot 3 \cdot 0.7 \cdot 1 \cdot 10.7) + 0.7^3 = 225.043 > 218.216$$

$$\begin{array}{r|ccccccccccc}
 & 1 & 0 & \cdot & 6 & & & & & & & \\
\cline{2-12}
1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
 & 1 & & & & & & & & & & \\
\cline{2-9}
10.6 & 0 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & & & \\
 & & 1 & 9 & 1 & \cdot & 0 & 1 & 6 & & & \\
\cline{2-9}
 & & & 2 & 7 & \cdot & 2 & 0 & 0 & & &
\end{array}$$

Choose $c_2 = 0.6$. Set $q = 10.6$, remainder $r = 218.216 - 191.016 = 27.200$.

5. **Bring down the next group**: Bring the next group 100 and append to $r = 27.200$, forming $n' = 27.200100$.

$$\begin{array}{r|ccccccccccc}
 & 1 & 0 & \cdot & 6 & & & & & & & \\
\cline{2-12}
1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
 & 1 & & & & & & & & & & \\
\cline{2-9}
10.6 & 0 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & & & \\
 & & 1 & 9 & 1 & \cdot & 0 & 1 & 6 & & & \\
\cline{2-12}
 & & & 2 & 7 & \cdot & 2 & 0 & 0 & 1 & 0 & 0
\end{array}$$

6. **Determine the next candidate**: With $n' = 27.200100$ and $q = 10.6$, find the largest c_3 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 27.200100$

   - If $c_3 = .07$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (10.6 \cdot 10^0) + .07 = 10.67$,
   $$f(.07, 10.6, 10.67) = (10^0 \cdot 3 \cdot 0.07 \cdot 10.6 \cdot 10.67) + 0.07^3 = 23.751763 \leq 27.200100$$
   - If $c_3 = .08$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (10.6 \cdot 10^0) + .08 = 10.68$,
   $$f(.08, 10.6, 10.68) = (10^0 \cdot 3 \cdot 0.08 \cdot 10.6 \cdot 10.68) + 0.08^3 = 27.170432 \leq 27.200100$$
   - If $c_3 = .09$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (10.6 \cdot 10^0) + .09 = 10.69$,
   $$f(.09, 10.6, 10.69) = (10^0 \cdot 3 \cdot 0.09 \cdot 10.6 \cdot 10.69) + 0.09^3 = 30.595509 > 27.200100$$

$$\begin{array}{r|ccccccccccc}
 & 1 & 0 & \cdot & 6 & 8 & & & & & & \\
\cline{2-12}
1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
 & 1 & & & & & & & & & & \\
\cline{2-9}
10.6 & 0 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & & & \\
 & & 1 & 9 & 1 & \cdot & 0 & 1 & 6 & & & \\
\cline{2-12}
10.68 & & & 2 & 7 & \cdot & 2 & 0 & 0 & 1 & 0 & 0 \\
 & & & 2 & 7 & \cdot & 1 & 7 & 0 & 4 & 3 & 2 \\
\cline{2-12}
 & & & & 0 & \cdot & 0 & 2 & 9 & 6 & 6 & 8 \\
\cline{2-12}
\end{array}$$

Choose $c_3 = .08$. Set $q = 10.68$, remainder $r = 27.200100 - 27.170432 = 0.029668$.

7. **Iterate or terminate**: No groups remaining and $r = 0.029668 \neq 0$. We can bring more group to find the more precisions, But will stop here with 2 decimals.

8. **Output**: Thus, $\sqrt[3]{1218.2161} \approx 10.680$, computed by the long division method.

3.3.3 Flexible Grouping: $1\,|\,218.2161$

In this example,We partition the digits as follows, and group the partitions as below:

$$\underbrace{\underbrace{1}_{\text{Partition 1}}}_{\text{Group 1}} \;\Big|\; \underbrace{\underbrace{218}_{\text{Partition 2}} \;\Big|\; \underbrace{.216}_{\text{Partition 3}} \;\Big|\; \underbrace{100}_{\text{Partition 4}}}_{\text{Group 2}}$$

1. **Group the digits**: Split 1218.2161 into partition by digit partition rule and group into computational units: $1\,|\,218.2161$ (or $001\,|\,218.216100$, where the leftmost & rightmost group is padded as needed).

2. **Initialize with the leftmost group**: With $n' = 1$ and $q = 0$, find the largest c_1 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 1$

   - If $c_1 = 0$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 0 = 0$,
   $$f(0, 0, 0) = (10^1 \cdot 3 \cdot 0 \cdot 0 \cdot 0) + 0^3 = 0 \leq 1$$
   - If $c_1 = 1$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 1 = 1$,
   $$f(1, 0, 1) = (10^1 \cdot 3 \cdot 1 \cdot 0 \cdot 1) + 1^3 = 1 \leq 1$$
   - If $c_1 = 2$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^1) + 2 = 2$,
   $$f(2, 0, 2) = (10^1 \cdot 3 \cdot 2 \cdot 0 \cdot 2) + 2^3 = 8 > 1$$

$$\begin{array}{r|ccccccccccc}
 & 1 \\ \cline{2-12}
1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
 & 1 \\ \cline{2-2}
 & 0
\end{array}$$

   Choose $c_1 = 1$. Set $q = 1$, remainder $r = 1 - 1 = 0$.

3. **Bring down the next group**: Bring the next group 218.216100 and append to $r = 0$, forming $n' = 218.216100$.

$$\begin{array}{r|ccccccccccc}
 & 1 \\ \cline{2-12}
1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
 & 1 \\ \cline{2-12}
 & 0 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0
\end{array}$$

4. **Determine the next candidate**: With $n' = 218.216100$ and $q = 1$, find the largest c_2 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 218.216100$

   - If $c_2 = 0.67$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1 \cdot 10^1) + 0.67 = 10.67$,
   $$f(0.67, 1, 10.67) = (10^1 \cdot 3 \cdot 0.67 \cdot 1 \cdot 10.67) + 0.67^3 = 214.767763 \leq 218.216100$$
   - If $c_2 = 0.68$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1 \cdot 10^1) + 0.68 = 10.68$,
   $$f(0.68, 1, 10.68) = (10^1 \cdot 3 \cdot 0.68 \cdot 1 \cdot 10.68) + 0.68^3 = 218.186432 \leq 218.216100$$
   - If $c_2 = 0.69$, then $k = 1$ and $q' = (q \cdot 10^k) + c = (1 \cdot 10^1) + 0.69 = 10.69$,
   $$f(0.69, 1, 10.69) = (10^1 \cdot 3 \cdot 0.69 \cdot 1 \cdot 10.69) + 0.69^3 = 221.611509 > 218.216100$$

$$\begin{array}{r|ccccccccccc}
 & 1 & 0 & \cdot & 6 & 8 \\ \cline{2-12}
1) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
 & 1 \\ \cline{2-12}
10.68 & 0 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
 & & 2 & 1 & 8 & \cdot & 1 & 8 & 6 & 4 & 3 & 2 \\ \cline{2-12}
 & & & & 0 & . & 0 & 2 & 9 & 6 & 6 & 8 \\ \cline{2-12}
\end{array}$$

   Choose $c_2 = 0.68$. Set $q = 10.68$, remainder $r = 218.216100 - 218.186432 = 0.029668$.

5. **Iterate or terminate**: No groups remaining and $r = 0.029668 \neq 0$. We can bring more group to find the more precisions, But will stop here with 2 decimals.

6. **Output**: Thus, $\sqrt[3]{1218.2161} \approx 10.68$, computed by the long division method.

3.3.4 Flexible Grouping: $1218.216\,|\,1$

In this example,We partition the digits as follows, and group the partitions as below:

$$
\underbrace{\underbrace{1}_{\text{Partition 1}} \quad | \quad \underbrace{218}_{\text{Partition 2}} \quad | \quad \underbrace{.216}_{\text{Partition 3}}}_{\text{Group 1}} \ \underbrace{\underbrace{100}_{\text{Partition 4}}}_{\text{Group 2}}
$$

1. **Group the digits**: Split 1218.2161 into partition by digit partition rule and group into computational units: $1218.216\,|\,1$ (or $001218.216\,|\,100$, where the leftmost & rightmost group is padded as needed).
2. **Initialize with the leftmost group**: With $n' = 1218.216$ and $q = 0$, find the largest c_1 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 1218.216$
   - If $c_1 = 10.5$, then $k = 2$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^2) + 10.5 = 10.5$,
   $$f(10.5, 0, 10.5) = (10^2 \cdot 3 \cdot 10.5 \cdot 0 \cdot 10.5) + 10.5^3 = 1157.625 \leq 1218.216$$
   - If $c_1 = 10.6$, then $k = 2$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^2) + 10.6 = 10.6$,
   $$f(10.6, 0, 10.6) = (10^2 \cdot 3 \cdot 10.6 \cdot 0 \cdot 10.6) + 10.6^3 = 1191.016 \leq 1218.216$$
   - If $c_1 = 10.7$, then $k = 2$ and $q' = (q \cdot 10^k) + c = (0 \cdot 10^2) + 10.7 = 10.7$,
   $$f(10.7, 0, 10.7) = (10^2 \cdot 3 \cdot 10.7 \cdot 0 \cdot 10.7) + 10.7^3 = 1225.043 > 1218.216$$

   $$
   \begin{array}{r|ccccccccccc}
    \multicolumn{1}{r}{} & 1 & 0 & \cdot & 6 & & & & & & & \\ \cline{2-12}
   10.6) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
    \multicolumn{1}{r}{} & 1 & 1 & 9 & 1 & \cdot & 0 & 1 & 6 & & & \\ \cline{2-9}
    \multicolumn{1}{r}{} & & & 2 & 7 & \cdot & 2 & 0 & 0 & & &
   \end{array}
   $$

   Choose $c_1 = 10.6$. Set $q = 10.6$, remainder $r = 1218.216 - 1191.016 = 27.200$.
3. **Bring down the next group**: Bring the next group 100 and append to $r = 27.200$, forming $n' = 27.200100$.

   $$
   \begin{array}{r|ccccccccccc}
    \multicolumn{1}{r}{} & 1 & 0 & \cdot & 6 & & & & & & & \\ \cline{2-12}
   10.6) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
    \multicolumn{1}{r}{} & 1 & 1 & 9 & 1 & \cdot & 0 & 1 & 6 & & & \\ \cline{2-12}
    \multicolumn{1}{r}{} & & & 2 & 7 & \cdot & 2 & 0 & 0 & 1 & 0 & 0
   \end{array}
   $$

4. **Determine the next candidate**: With $n' = 27.200100$ and $q = 10.6$, find the largest c_2 such that $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3 \leq 27.200100$
   - If $c_2 = .07$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (10.6 \cdot 10^0) + .07 = 10.67$,
   $$f(.07, 10.6, 10.67) = (10^0 \cdot 3 \cdot 0.07 \cdot 10.6 \cdot 10.67) + 0.07^3 = 23.751763 \leq 27.200100$$
   - If $c_2 = .08$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (10.6 \cdot 10^0) + .08 = 10.68$,
   $$f(.08, 10.6, 10.68) = (10^0 \cdot 3 \cdot 0.08 \cdot 10.6 \cdot 10.68) + 0.08^3 = 27.170432 \leq 27.200100$$
   - If $c_2 = .09$, then $k = 0$ and $q' = (q \cdot 10^k) + c = (10.6 \cdot 10^0) + .09 = 10.69$,
   $$f(.09, 10.6, 10.69) = (10^0 \cdot 3 \cdot 0.09 \cdot 10.6 \cdot 10.69) + 0.09^3 = 30.595509 > 27.200100$$

   $$
   \begin{array}{r|ccccccccccc}
    \multicolumn{1}{r}{} & 1 & 0 & \cdot & 6 & & & & & & & \\ \cline{2-12}
   10.6) & 1 & 2 & 1 & 8 & \cdot & 2 & 1 & 6 & 1 & 0 & 0 \\
    \multicolumn{1}{r}{} & 1 & 1 & 9 & 1 & \cdot & 0 & 1 & 6 & & & \\ \cline{2-12}
    \multicolumn{1}{r}{} & & & 2 & 7 & \cdot & 2 & 0 & 0 & 1 & 0 & 0 \\
    \multicolumn{1}{r}{} & & & 2 & 7 & \cdot & 1 & 7 & 0 & 4 & 3 & 2 \\ \cline{2-12}
    \multicolumn{1}{r}{} & & & & 0 & \cdot & 0 & 2 & 9 & 6 & 6 & 8 \\ \cline{2-12}
   \end{array}
   $$

   Choose $c_2 = .08$. Set $q = 10.68$, remainder $r = 27.200100 - 27.170432 = 0.029668$.
5. **Iterate or terminate**: No groups remaining and $r = 0.029668 \neq 0$. We can bring more group to find the more precisions, But will stop here with 2 decimals.
6. **Output**: Thus, $\sqrt[3]{1218.2161} \approx 10.68$, computed by the long division method.

4 Discussion

The method presented here delivers a single, unified long division algorithm for extracting cube roots of any real number—whole, decimal, perfect, or imperfect. This stands in contrast to existing public long division schemes, which typically handle only specific cases or require ad hoc rules for non-integers. By supporting flexible partition grouping and presenting an explicit generalized formula, this approach eliminates the piecemeal, case-specific logic seen in earlier methods.

The key advantage, and the very reason for this algorithm's existence, is its suitability for exact, high-precision manual computation. Unlike Newton's method—which excels in digital computation but lacks intuitive intermediate steps for hand calculation—this algorithm allows full control over precision, with each iteration yielding string-length-preserving, readable results. This explicitness makes it particularly valuable in educational settings, where procedural transparency trumps mere approximation.

In practical terms, this method serves teachers and students needing an exact, step-by-step procedure for cube root extraction, especially when calculators or software are forbidden. Its transparency also benefits anyone seeking to understand the mechanics behind root extraction, not just the final answer.

Limitations are honest and clear: while the procedure applies to all real numbers, it does not address complex roots—a realm well outside the intent and reach of this work. Manual calculation of very large or very small numbers can become tedious, though the method itself does not fail or degrade in precision. Earlier algorithm versions failed at decimal cases, but the approach here closes that gap with full generality for the real numbers. There may be faster or more efficient manual approaches for specific contexts, but such alternatives were not the focus here.

One incidental insight gained while developing this method is the remarkable consistency of patterns underlying arithmetic partitioning—suggesting that even complex-seeming calculations are ultimately governed by discoverable, generalized rules.

Future work could explore systematic extension to higher-order roots or broader classes of real functions via long division-style partitioning. However, the current scope is sharply focused: a definitive, teachable, and precise method for real cube root extraction by long division.

5 Conclusion

This paper presents a unified long division algorithm for cube root extraction that operates consistently across all real numbers. The method's core contribution lies in its generalized remainder formula $f(c, q, q') = (10^k \cdot 3 \cdot c \cdot q \cdot q') + c^3$, which adapts seamlessly to integer, decimal, perfect, and non-perfect cubes through dynamic parameter adjustment. For negative inputs, the algorithm applies directly to the absolute value, with the result's sign determined by the radicand (e.g., $\sqrt[3]{-27} = -3$), extending applicability to all real numbers without modification.

Unlike traditional digit-by-digit methods constrained to single-step extraction, this algorithm supports flexible partition grouping — bringing down one, two, or more partitions per iteration. This grouping mechanism provides computational control: smaller groups prioritize precision through granular steps, while larger groups accelerate convergence at the cost of broader candidate searches. The formula's behavior remains algebraically consistent regardless of group size or number type.

The worked examples demonstrate practical application across the full spectrum of cases: whole number perfect cubes, decimal inputs, and arbitrary-precision non-perfect cubes. By treating candidates as string-length-preserving values (maintaining trailing zeros in k calculation), the method avoids case-specific branching while guaranteeing correct positional scaling through q' updates.

This approach fills a gap in manual computation techniques, offering an intuitive, teachable alternative to iterative approximation methods. Its systematic structure makes it suitable for educational contexts where understanding the mechanics of root extraction matters, while its arbitrary precision capability ensures practical utility for hand calculations requiring exact or high-accuracy results.

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Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

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